The control system designer has the problem of selecting a configuration using the information in Table 1. One approach to this problem is to construct another function from the table that indicates how the system will most probably perform during a mission. This requires modeling the failure characteristics of the control system components as a function of mission time t. For this example, the failure characteristics are assumed to be identical for each component and described by an exponential probability density function for the time of failure as a function of mission time,

$$p(t) = \exp(-t/t_m)/t_m$$

It is suggested that the designer should weight J^* of Table 1 according to the probability that the mode will occur at the end of the mission. This has been done and Fig. 2 shows the variation of this weighted performance indicator as a function of t/t_M. Note that at $t/t_M = 0$, the weighted cost function is equal to that in Table 1 for the mode with no failures. Also, note that the weighted cost goes to zero as t/t_M increases. This occurs because the set of failure modes considered by the designer is not complete. It is possible that three, four, or even five simultaneous failures may occur. These are not considered in the design set so as not to bias the selection of actuator configuration with catastrophic failure modes. Figure 3 is a graph of the incremental cost of configuration 2 over configuration 1. The notable point is that configuration 2 is preferable to 1 for short missions, whereas configuration 1 is preferable to 2 for long missions.

Conclusions

This Note has described a methodology that allows the control system designer to select from among a set of possible actuator locations the one that is best for vibration suppression considering the reliability of the components. This extends the results of earlier work that was developed using static modeling for the shape control problem to problems that involve the dynamics of a flexible spacecraft. The method has been applied to a grid structure as an example. In that case, it was shown that optimal locations depend on the design mission life. The method generally involves, for each candidate location set, determining the best achievable performance of the control system for all failure modes that the designer wishes to consider. These values of performance are then used to construct the criterion function by taking the probability-weighted sum of the performance measures for each failure mode. The actuator locations are then chosen to minimize this cost criterion.

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Subaliases in the Frequency Response of Digitally Controlled Systems

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I. Introduction

In a recent paper, Whitbeck, Didaleusky, and Hofmann[1] extended the concept of the traditional "sampled-spectrum" frequency response for discretely excited continuous systems. When a sinusoidal wave is input to a discretely excited continuous system, N sine waves at different alias frequencies are required to match the continuous steady-state time response at the sampling instants and at $N-1$ equally spaced intersample points. In the special case of $N=1$, this reduces to the traditional concept of the sampled-spectrum frequency response for sampled-data systems. Letting N approach infinity gives an infinite spectrum for the continuous steady-state response of a discretely excited continuous system. This theory enables one to write an exact expression of the time response sampled at any rate that is an integer multiple of the sampling rate of the system. The practical value of knowing such an expression of the output is evident.

However, in the derivation of Whitbeck, Didaleusky, and Hofmann,[1] only positive aliases of the input frequency are included in the representation of the sampled continuous output. This result is correct for the case where N is finite, but contains only half of the spectral components necessary for an asymptotic representation of the continuous output for the limit case where N approaches infinity. In this paper, a new derivation, which includes subaliases in the spectral representation of the sampled output, will be presented. It will be shown that, as the output sampling rate N approaches infinity, the infinite spectrum of the continuous output contains all aliases and subaliases of the input frequency. Also presented will be a direct derivation of the infinite spectrum of the continuous output, without invoking the expression of the sampled steady-state response. This confirms the correct representation of the continuous steady-state output of discretely excited continuous systems in response to a single sinusoidal input. This derivation brings forth a unified concept of frequency response, which enables one to write the spectral representation of the output of a discretely excited continuous system on the basis of the frequency response of the continuous system.

II. Frequency Components in the Sampled Output

Consider the system of Fig. 1, where $G(s)$ represents an arbitrary transfer function and $M(s)$ represents an arbitrary data hold. Let the input be a unit amplitude exponential ϵ^{jbt} and the output be sampled with period T/N. Using multirate sampling results (see Appendix of Ref. 1) yields

$$C^{T/N} = [GMR^T]^{T/N} = (GM)^{T/N}R^T$$

$$= (GM)^{T/N}\frac{z^N}{z^N - \epsilon^{jbT}} \qquad (z \triangleq \epsilon^{sT/N}) \tag{1}$$

where the notation follows Ref. 1 and the superscript denotes the period of sampling operation. In the time domain, a sampled function $[r(t)]^T$ or r^T is defined by

$$[r(t)]^T \triangleq \sum_{n=-\infty}^{\infty} r(n)\delta(t-nT) = r(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \tag{2}$$

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In the s domain, a sampled function R^T is defined by

$$R^T(s) \triangleq \sum_{n=0}^{\infty} r(n)\epsilon^{-nsT} = \frac{1}{T}\sum_{n=-\infty}^{\infty} R\left(s+j\frac{2\pi n}{T}\right) \tag{3}$$

and in the z domain by

$$R^T(z) \triangleq R^T(s)\,|_{\epsilon^{sT}\to z} \tag{4}$$

However, in the presence of a higher rate sampler with period T/N, as in this paper, it is desirable to use $z=\epsilon^{sT/N}$ and

$$R^T(z^N) \triangleq R^T(s)\,|_{\epsilon^{sT/N}\to z} \tag{5}$$

Hence it is simpler to think of frequency response of discretely excited continuous systems in terms of s rather than z. Where no confusion may arise, the variables s and z may be omitted to give versatility to the notation. For the sake of notational brevity in the following development $G(s)M(s)$ will be written as $GM(s)$, for occasions where s needs to be substituted by a string of notations.

The steady-state component in the sampled output is of interest. In taking the partial fraction expansion of the right-hand side of Eq. (1), one may note that the N principal roots of $z^N-\epsilon^{jbT}$ are

$$\epsilon^{jbT/N},\quad \epsilon^{j(bT-2\pi)/N},\quad \epsilon^{j(bT+2\pi)/N},\quad \epsilon^{j(bT-4\pi)/N},\ldots$$

Hence the partial fraction expansion of $(1/z)C^{T/N}(z)$ may be written as

$$\frac{1}{z}C^{T/N}(z) = \sum_{n=n_1}^{n_2}\frac{A_n+jB_n}{z-\epsilon^{j\omega_n T/N}} + [\text{terms due to modes of } (GM)^{T/N}] \tag{6}$$

where

$$\omega_n = b+\frac{2\pi n}{T}$$

$$\left.\begin{aligned} n_1 &= \frac{-1}{2}(N-1)\\ n_2 &= \frac{1}{2}(N-1)\end{aligned}\right\}\text{ if } N=\text{odd} \qquad \left.\begin{aligned} n_1 &= \frac{-N}{2}\\ n_2 &= \frac{N}{2}-1\end{aligned}\right\}\text{ if } N=\text{even}$$

and

$$A_n+jB_n = \frac{(GM)^{T/N}z^{N-1}}{\mathrm{d}/\mathrm{d}z(z^N-\epsilon^{jbT})}\bigg|_{z=\epsilon^{j\omega_n T/N}} = \frac{1}{N}(GM)^{T/N}\bigg|_{z=\epsilon^{j\omega_n T/N}}$$

$$= \frac{1}{N}(GM(s))^{T/N}\bigg|_{s=j\omega_n} \tag{7}$$

The steady-state response to ϵ^{jbt} can be written by inspection of Eq. (6) as

$$[c_{ss}(t)]^{T/N} = \left[\sum_{n=n_1}^{n_2}(A_n+jB_n)\epsilon^{j\omega_n t}\right]^{T/N} \tag{8}$$

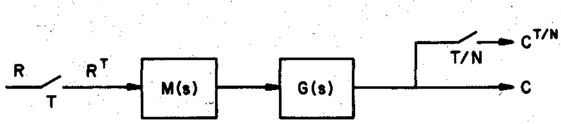

Fig. 1 Discretely excited continuous system.

Since the system of Fig. 1 is linear, the response to the imaginary part of ϵ^{jbt} is the imaginary part of the response to ϵ^{jbt}. Hence the steady-state response to the $\sin bt$ is

$$[c_{ss}(t)]^{T/N}\left[\sum_{n=n_1}^{n_2}(A_n\sin\omega_n t+B_n\cos\omega_n t)\right]^{T/N} \tag{9}$$

where A_n and B_n are determined by Eq. (7). In the right-hand side of Eq. (9), the N spectral components are selected alternately between negative (subalias) and positive aliases, while only positive aliases are used in Eq. (11) of Ref. 1. It is worth noting that while both results are correct for the representation of sampled steady-state output, difficulty arises in considering the limit case of $N\to\infty$ in Eq. (11) of Ref. 1. The following development demonstrates this point.

Letting $N\to\infty$ in Eq. (9) gives

$$c_{ss}(t) = \sum_{n=-\infty}^{\infty}(A_n\sin\omega_n t+B_n\cos\omega_n t) \tag{10}$$

where, because of Eq. (7),

$$A_n+B_n = \frac{1}{T}\lim_{n\to\infty}\frac{T}{N}(GM)^{T/N}\bigg|_{s=j\omega_n}$$

$$= \frac{1}{T}\lim_{N\to\infty}\sum_{k=-\infty}^{\infty}GM\left(s+j\frac{2\pi kN}{T}\right)\bigg|_{s=j\omega_n} \tag{11}$$

As long as $-N/2\leq n_1\leq n\leq n_2<N/2$ and $N\to\infty$, only the $k=0$ term contributes to the infinite sum in the right-hand side of Eq. (11) as $GM(s)$ is always a low-pass filter. Hence, as $N\to\infty$,

$$A_n+jB_n = \frac{1}{T}GM(j\omega_n) \tag{12}$$

However, the above limit case is not true if $n=N-i$, for finite integer i. In that case,

$$A_n+jB_n = \frac{1}{T}\lim_{N\to\infty}\sum_{k=-\infty}^{\infty}GM\left(s-j\frac{2\pi kN}{T}\right)\bigg|_{s=j\omega_n}$$

$$= \frac{1}{T}\lim_{N\to\infty}\sum_{k=-\infty}^{\infty}GM\left[j\left(b+\frac{2\pi(n-kN)}{T}\right)\right]$$

$$= \frac{1}{T}\lim_{N\to\infty}\sum_{k=-\infty}^{\infty}GM\left[j\left(b-\frac{2\pi i}{T}\right)-j\frac{2\pi(k-1)N}{T}\right] \tag{13}$$

Now as $N\to\infty$, only the $k=1$ term contributes to the infinite sum in the right-hand side of Eq. (13) as $GM(s)$ is always a low-pass filter. Thus, for $n=N-i$, $N\to\infty$, finite i,

$$A_n+jB_n = \frac{1}{T}GM\left[j\left(b-\frac{2\pi i}{T}\right)\right] \tag{14}$$

which is a subalias component, is not negligible, and is not represented by Eq. (12). Since the summation in Eq. (11) of Ref. 1 runs between $n=0$ and $n=N-1$, letting $N\to\infty$ results in an A_n+jB_n which cannot be represented by Eq. (12), which is Eq. (18) of Ref. 1.

III. Direct Derivation of the Frequency Spectrum of the Continuous Output

In the preceding development, the continuous output is treated as a limit case of sampled output with an arbitrarily large sampling frequency, and the infinite frequency spectrum of the continuous output is given by Eq. (10) with the spectral components $A_n + jB_n$ given by Eq. (12). In this section, the frequency spectrum of the continuous output of a discretely excited continuous system will be derived directly without invoking the sampled output. Thus, it confirms the result of the previous section.

Again consider the system of Fig. 1 where

$$C(s) = G(s)M(s)R^T(s) \quad (15)$$

Let

$$g_m(t) \triangleq \mathcal{L}^{-1}[G(s)M(s)] \quad (16)$$

Let the input be e^{jbt} with t extending from $-\infty$ to ∞. Then the steady-state response of the system may be written as

$$c_{ss}(t) = [e^{jbt}]^{T} * g_m(t) \quad (17)$$

where the asterisk denotes convolution, and

$$[e^{jbt}]^T = e^{jbt} \sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T} e^{jbt} \sum_{n=-\infty}^{\infty} e^{j(2\pi n/T)t} \quad (18)$$

The last identity can be found in textbooks on Fourier series and Fourier transform.[2] Fourier transformation of Eq. (18) gives

$$\mathfrak{F}[(e^{jbt})^T] = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - b - \frac{2\pi n}{T}\right) \quad (19)$$

Fourier transform of Eq. (17) is

$$C_{ss}(j\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - b - \frac{2\pi n}{T}\right) G(j\omega)M(j\omega) \quad (20)$$

since $g_m(t)$ is a casual function. Now the steady-state output is obtained by inverse transforming Eq. (20):

$$c_{ss}(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} GM\left[j\left(b + \frac{2\pi n}{T}\right)\right] e^{j[b+(2\pi n/T)]t} \quad (21)$$

Since $\omega_n = b + \frac{2\pi n}{T}$, substituting Eq. (12) into Eq. (21) gives

$$c_{ss}(t) = \sum_{n=-\infty}^{\infty} (A_n + jB_n) e^{j\omega_n t} \quad (22)$$

as the steady-state response to e^{jbt}. Note that Eq. (22) is the limit case of Eq. (8) when $N \to \infty$. Again, by virtue of the linearity of the system, Eq. (22) implies that the steady-state response to sinbt is given by Eq. (10), with $A_n + jB_n$ given by Eq. (12).

Acknowledgments

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